Applications of Stochastic Processes and Fractional Calculus in Plasma Physics

MAHMOUD EMADELDIN MAHMOUD

Mansoura University

April 24, 2025

Outline

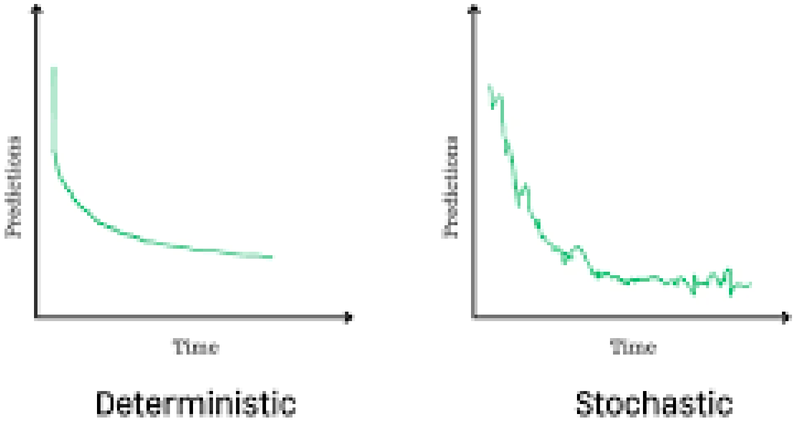
Predictions
Time
Deterministic
Predictions
Time
Stochastic

Figure: 1

Stochastic Processes

Definition

A **stochastic process** (or random process) is the time evolution of a random variable.

- is very important both in mathematical theory and its applications in engineering, economics, Biology, Physics, Chemistry, ecology and etc.
- It is used to model a large number of various phenomena where the quantity of interest varies discretely or continuously through time in a non-predictable fashion. e.g, noisy phenomena, fluctuation, and probabilistic behavior

Properties

- **Stationarity:** Statistical properties are constant over time shifts.
- **Filtration:** Represents the flow of information available over time.
- **Modification:** Two processes agreeing at any single time point with probability 1.
- **Indistinguishable:** Two processes having identical entire sample paths with probability 1.
- **Separability:** Properties determined by a countable dense subset of time points.
- **Independence:** Knowing one process gives no information about the other.
- **Uncorrelatedness:** Zero cross-covariance between the processes for all time pairs.
- **Orthogonality:** Zero cross-correlation between the processes for all time pairs

Properties (2)

- **Markovianity:**The future evolution of the process depends only on its current state, not on its entire past history.
- **Ergodicity:**Time averages calculated along a single, long sample path converge to the ensemble averages Gaussianity
- **Gaussianity:** Any finite collection of random variables from the process follows a multivariate Gaussian (normal) distribution
- **Martingale Property:** The conditional expectation of the future value, given the history up to the present, is equal to the present value.

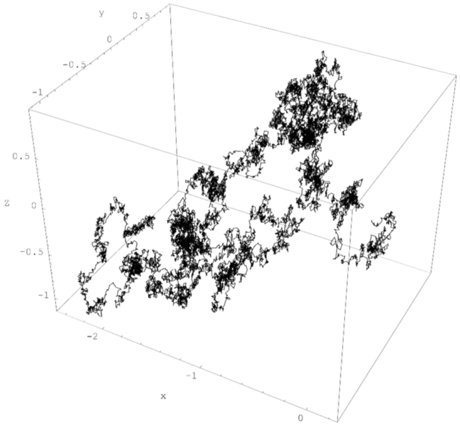

Figure: 1

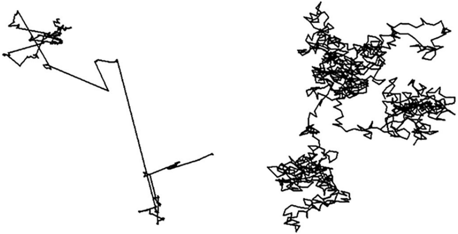

Figure: 3

Stochastic Processes Types

- **Bernoulli Process:** a sequence of independent and identically distributed (iid) random variables, where each random variable takes either the value one or zero
- **Poisson Process:** .
- **Wiener Process:**
- **Gaussian Processes:**
- **Martingale Processes:**
- **Lévy process**
- **renewal processes**
- **branching processes**

Stochastic Processes Modelling

$$p(x_3, t_3|x_1, t_1) = \int_{-\infty}^{\infty} p(x_3, t_3|x_2, t_2)\, p(x_2, t_2|x_1, t_1)\, dx_2 \tag{1}$$

$$\frac{dP_n(t)}{dt} = \sum_{m \neq n} [W_{mn} P_m(t) - W_{nm} P_n(t)] \tag{2}$$

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[A(x,t)P(x,t)] + \frac{\partial^2}{\partial x^2}[D(x,t)P(x,t)] \tag{3}$$

$$\frac{dx(t)}{dt} = v(t) \tag{4}$$

$$m\frac{dv(t)}{dt} = -\zeta v(t) + F(x(t), t) + \xi(t) \tag{5}$$

Fractional Calculus

$$\frac{d^{1/2} f}{dx^{1/2}} = ?$$

Figure: A fractional Derivative ?

Definition

fractional derivative is a generalization of the familiar concept of differentiation to non-integer orders. α or a transformation from space to another space

- **Memory Effects:** Systems where the future state depends on the entire history, not just the present (non-Markovian).
- **Non-locality:** Interactions or transport that are not point-like but depend on conditions in a surrounding region.
- **Anomalous Scaling:** Processes where variance doesn't scale linearly with time ($\langle x^2 \rangle \sim t^\alpha$, $\alpha \neq 1$). Common in complex, heterogeneous, or crowded media.
- **Power-law Behavior:** Capturing phenomena exhibiting power-law frequency or time dependencies (e.g., viscoelastic materials)

Riemann-Liouville (RL) Fractional Integral

For $\alpha > 0$, the RL fractional integral of order α is:

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau $$

where $\Gamma(\alpha)$ is the Gamma function. Captures weighted history of $f(t)$.

Riemann-Liouville (RL) Fractional Derivative

For $n-1 < \alpha < n$,

$$ {}_aD_t^{\alpha} f(t) = \frac{d^n}{dt^n} \left({}_aI_t^{n-\alpha} f(t) \right) $$

$$ = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau $$

Properties

- **Linearity:** Fractional operators are linear.
- **Non-locality / Memory:** The value $D^{\alpha}f(t)$ depends on $f(\tau)$ for $\tau \in [a, t]$.
- **Composition Law:** Generally $D^{\alpha}D^{\beta}f \neq D^{\alpha+\beta}f$
- **Laplace Transform (Caputo, $a = 0$):** For $n - 1 < \alpha \leq n$,

$$\mathcal{L}\{{}_0^C D_t^{\alpha} f(t); s\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0^+)$$

Applications

- **Physics:** Viscoelasticity, anomalous diffusion, chaos theory, wave propagation in complex media.
- **Engineering:** Control theory (designing controllers for systems with memory, potentially leading to better performance),
- **signal processing:**(filtering, modeling 1/f noise),
- **bioengineering:**(modeling biological tissues, drug delivery).
- **Finance:** Modeling long-range dependence in financial markets (e.g., volatility clustering).
- **Biology:** Modeling dynamics in biological systems that exhibit memory, like cell membrane mechanics or population dynamics.

Fractional Calculus in Plasma: Anomalous Transport Models of Energetic Particles In Space Plasma

- Particle transport in turbulent magnetic fields in space plasmas is considered to be subdiffusive perpendicular to the mean magnetic field.
- The evidence for that is the data collected by spacecraft (e.g., Ulysses and Voyager) which manifest that the transport of energetic particles in the turbulent heliospheric medium can indeed be super-diffusive
- Some endeavors to understand the energetic particle transport in the cosmos consider the Langevin equation [?, ?] for the coordinate $z(t)$

$$\frac{dv}{dt} = -\eta v + \frac{F(z)}{m} + \xi(t), \quad \frac{dz}{dt} = v. \tag{6}$$

$$\frac{\partial W(z,v,t)}{\partial t} = \left[-v\frac{\partial}{\partial z} + \frac{\partial}{\partial v}\left(\eta v - \frac{F(z)}{m}\right) + \frac{\eta k_B T}{m}\frac{\partial^2}{\partial v^2}\right] W(z,v,t). \tag{7}$$

after some assumption it becomes

$$\frac{\partial}{\partial t} f(v,t) = \eta \frac{\partial}{\partial v}(v f(v,t)) + A \frac{\partial^2}{\partial v^2} f(v,t), \tag{8}$$

in order to investigate solutions describing the sub-diffusive and super-diffusive behaviors relevant for the propagation of energetic particles in space plasmas, we start by considering the space-time fractional force-less Fokker–Planck equation in the form:

$$^{C}D_t^{\alpha} f(v,t) = \eta \frac{\partial}{\partial v}(v f(v,t)) + A \frac{\partial^{\beta}}{\partial |v|^{\beta}} f(v,t), \tag{9}$$

To solve Eq (9), consider a separation ansatz of the form

$$f(v,t) = T(t)\varphi(v). \tag{10}$$

$$\frac{1}{T(t)} {}^{C}D_t^{\alpha} T(t) = \frac{1}{\varphi(v)} \left(\eta \frac{\partial}{\partial v}(v\varphi(v)) + A \frac{\partial^{\beta}\varphi(v)}{\partial |v|^{\beta}} \right). \tag{11}$$

The two sides in Eq. (11) depend on two variables and correspond to a constant λ. Then,

$$ {}^{C}D_t^{\alpha} T(t) = \lambda T(t), \tag{12} $$

$$ \eta \frac{\partial}{\partial v}(v\varphi(v)) + A \frac{\partial^{\beta}\varphi(v)}{\partial |v|^{\beta}} = \lambda \varphi(v), \tag{13} $$

and consider the initial values given by

$$ T(0) = 1, \quad \dot{T}(0) = 0 \quad (\text{assuming } \alpha > 1 \text{ if } \dot{T} \text{ is needed}). \tag{14} $$

First, to get the solution of $T(t)$, take the Laplace transform of Eq. (12). Assuming $0 < \alpha \leq 1$ (so only $T(0)$ is needed):

$$ s^{\alpha}\tilde{T}(s) - s^{\alpha-1}T(0) = \lambda \tilde{T}(s), \tag{15} $$

Using $T(0) = 1$:

$$ \tilde{T}(s) = \frac{s^{\alpha-1}}{s^{\alpha} - \lambda}. \tag{16} $$

Taking the inverse Laplace transform yields

$$ T(t) = E_{\alpha}(\lambda t^{\alpha}), \tag{17} $$

Taking the Fourier transformation of equation (13), the solution will be an exp function and we will write in it the fox-H function to obtain the inverse Fourier of it then the solution which will be

$$\varphi(v) = c_1 \frac{1}{|v|} H_{3,3}^{2,1} \left[\frac{\beta\eta}{A} \left| \frac{A}{\eta v} \right|^{\beta} \middle| \begin{matrix} (1,1),(1,1),(1-\frac{\lambda}{\eta},\frac{\beta}{2}) \\ (1-\frac{\lambda}{\eta},\beta),(1,1),(1-\frac{\lambda}{2\eta},\frac{\beta}{2}) \end{matrix} \right]. \quad (18)$$

so the general solution will be

$$f(v,t) = \frac{E_\alpha(\lambda t^\alpha)}{|v|} H_{3,3}^{2,1} \left[\frac{\beta\eta}{A} \left| \frac{A}{\eta v} \right|^{\beta} \middle| \begin{matrix} (1,1),(1,1),(1-\frac{\lambda}{\eta},\frac{\beta}{2}) \\ (1-\frac{\lambda}{\eta},\beta),(1,1),(1-\frac{\lambda}{2\eta},\frac{\beta}{2}) \end{matrix} \right] ., 1 < \beta \leq$$

(19)

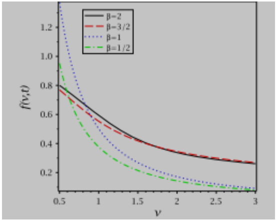

Figure: f(v,t)

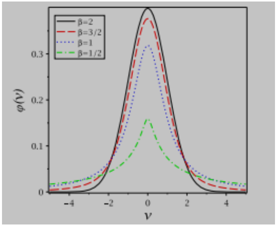

Figure: Q(v)

Langevin Equation for Dusty Plasma particle Temperature

A model is developed using the single-particle Langevin equation of motion to predict a particle temperature T. This temperature is an estimate of the true particle kinetic temperature. This model neglects microscopic collective fluctuations in the plasma. Heating is due to a combination of Brownian interaction with the neutral gas and electrostatic fluctuations, while cooling is due to neutral gas drag. The calculation of T is performed in analogy with the standard Langevin treatment for the Brownian motion of a particle in a viscous medium. The starting point for the calculation is the single- particle Langevin equation

$$m\frac{d^2x}{dt^2} = -m\omega_0^2 x - m\gamma\frac{dx}{dt} + \xi(t). \tag{20}$$

Our strategy is to solve Eq. (1) for the mean-square velocity$<v^2>$, which can then be used to compute the temperature, given by

$$T_L = m\langle v^2 \rangle \tag{21}$$

First we review the relationship between mean-square quantities, correlation functions, and power spectra. We define the Fourier transform pair for velocity as

$$\begin{aligned} v(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} v(\omega) e^{-i\omega t} d\omega, \\ v(\omega) &= \int_{-\infty}^{\infty} v(t) e^{i\omega t} dt. \end{aligned} \tag{22}$$

The velocity autocorrelation function is given by

$$C_{vv}(\tau) = \langle v(t)v(t+\tau) \rangle = \lim_{\theta \to \infty} \frac{1}{\theta} \int_{-\theta/2}^{\theta/2} v(t)v(t+\tau)dt. \tag{23}$$

The velocity power spectrum $G_v(\omega)$ and autocorrelation function are related by the Wiener-Khintchine relations:

$$C_{vv}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_v(\omega) e^{-i\omega\tau} d\omega,$$
$$G_v(\omega) = \int_{-\infty}^{\infty} C_{vv}(\tau) e^{i\omega\tau} d\tau. \qquad (24)$$

Using $v(t)$ from Eq. (22) in the integral of Eq. (23) and taking the Fourier transform yields

$$G_v(\omega) = \lim_{\theta\to\infty} \frac{1}{\theta}|v(\omega)|^2. \qquad (25)$$

We wish to derive an expression for $\langle v^2 \rangle$. This can be written as the velocity autocorrelation function evaluated at $\tau = 0$.

Using Eq. (24), $\langle v^2 \rangle$ can be expressed in terms of the power spectrum as

$$\langle v^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_v(\omega) d\omega. \tag{26}$$

Now we solve Eq. (20) for the velocity, obtaining

$$v(\omega) = \chi(\omega)\xi(\omega), \tag{27}$$

where

$$\chi(\omega) = \frac{-i\omega}{m(\omega_0^2 - \omega^2 + i\gamma\omega)} \tag{28}$$

is the response function. Using Eq. (25), the velocity power spectrum is

$$G_v(\omega) = |\chi(\omega)|^2 G_\xi(\omega), \tag{29}$$

where

$$G_\xi(\omega) = \lim_{\theta \to \infty} \frac{1}{\theta} |\xi(\omega)|^2 \tag{30}$$

is the power spectrum of the fluctuating force. Substituting this result into Eq. (26) yields the instantaneous mean-square velocity,

$$\langle v^2 \rangle = \frac{1}{2\pi m^2} \int_{-\infty}^{\infty} \frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} G_\xi(\omega) d\omega. \tag{31}$$

$$G_\xi(\omega) = G_\xi^{Br}(\omega) + G_\xi^{ES}(\omega). \tag{32}$$

In the absence of electrostatic fluctuations ($\xi_{ES} = 0$), the problem reduces to the usual treatment of Brownian motion [?, ?]. The particle temperature T_{Br} for Brownian motion is obtained from Eq. (31) by assuming the spectrum is flat, i.e., $G_\xi^{Br}(\omega)$ is constant for a frequency ω ranging from 0 to well above ω_0. This yields

$$T_{Br} = \frac{G_\xi^{Br}(0)}{2m\gamma}. \tag{33}$$

In analogy with the analysis for Brownian motion, we may now predict a temperature for a plasma crystal in the presence of electrostatic fluctuations. Using Eqs. (21), (31), and (32), the total particle temperature predicted by the Langevin model is

$$T_L = T_{Br} + T_{ES}, \tag{34}$$

where

$$T_{ES} = \frac{1}{2\pi m} \int_{-\infty}^{\infty} \frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} G_\xi^{ES}(\omega) d\omega \tag{35}$$

5. Conclusion

- **Stochastic Processes** are essential for describing plasma phenomena where randomness is inherent or emergent:
  - Collisions, wave-particle interactions, turbulence.
  - Tools: Fokker-Planck Eq., Langevin Eq., random walk models.
  - Capture fluctuations, diffusion, stochastic heating/acceleration.
- **Fractional Calculus** provides a powerful mathematical framework for systems with memory and non-locality:
  - Particularly relevant for **anomalous transport** (sub- and superdiffusion) in turbulent or complex plasmas.
  - Tools: Fractional kinetic equations (FFPE), fractional transport equations, fractional wave equations.
  - Captures power-law behaviors, trapping, long flights, non-local responses.
- Both approaches extend classical deterministic and diffusive models

References I

- Gardiner, C. W. (2009). *Stochastic Methods: A Handbook for the Natural and Social Sciences*. Springer.
- Van Kampen, N. G. (2007). *Stochastic Processes in Physics and Chemistry*. Elsevier.
- Risken, H. (1996). *The Fokker-Planck Equation: Methods of Solution and Applications*. Springer.
- Podlubny, I. (1998). *Fractional Differential Equations*. Academic Press.
- Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier.
- Chandrasekhar, S. (1943). Stochastic Problems in Physics and Astronomy. *Reviews of Modern Physics*, 15(1), 1-89.

References II

- Balescu, R. (2005). *Aspects of Anomalous Transport in Plasmas*. CRC Press.
- Rechester, A. B., & Rosenbluth, M. N. (1978). Electron heat transport in a stochastic magnetic field. *Physical Review Letters*, 40(1), 38-41.
- Zaslavsky, G. M. (2002). Chaos, fractional kinetics, and anomalous transport. *Physics Reports*, 371(6), 461-580.
- del-Castillo-Negrete, D. (2005). Fractional diffusion models of transport in fusion plasmas. *Physics of Plasmas*, 12(5), 052323.
- Sharma, R. K., & Garg, M. (2006). Fractional Fokker–Planck equation for plasma. *Physics Letters A*, 355(4-5), 285-288.
- Heuraux, S., et al. (Year). Title of relevant paper on fractional models in plasma waves/transport. *Journal Name*, Vol(Issue), pages.